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# SOME NOTES ON A FIBONACCI–LUCAS IDENTITY

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Abstract. In 2016, Edgar and, independently of him, Bhatnagar stated a nice polynomial identity that connects Fibonacci and Lucas numbers. Shortly after their publications, this identity was generalized in two different ways: Dafnis, Phillipou and Livieris provided a generalization to Fibonacci sequences of order $k$, while Abd-Elhameed and Zeyada extended the Edgar–Bhatnagar identity to generalized Fibonacci and Lucas sequences. In this paper, we present more polynomial identities for generalized Lucas sequences. We discuss interesting aspects and special cases which have not been stated before but deserve recognition. Finally, we prove the polynomial analogues of these identities for Chebyshev polynomials.

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## 1. Motivation

As usual, the Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ are defined, for $n \in \mathbb{Z}$, through the recurrence relations $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with initial values $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$, $L_1 = 1$. For negative subscripts we have $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$. They possess the explicit formulas (Binet forms)

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z}.$$

Standard references on these sequences are the textbooks by Koshy [8] and Vajda [20] in which a huge amount of additional information is presented.

Our motivation for these notes is the following prominent identity due to Edgar [7] and Bhatnagar [3] where they generalized two Fibonacci–Lucas identities from [9, 10]:

$$\sum_{k=0}^{n} x^k \big(L_k + (x-2)F_{k+1}\big) = x^{n+1}F_{n+1}, \tag{1}$$

which is referred to as Edgar–Bhatnagar identity.

This identity holds for all $x \in \mathbb{C}$ and the special cases $x = 2$ and $x = 3$, respectively, have been proved by Benjamin and Quinn [2] and Marques [10]. A polynomial variant was given by Sury [18]. Dafnis, Philippou and Livieris [5, 6, 13] have generalized this identity to Fibonacci and Lucas numbers of order $k$ and gave two different proofs for their generalization. In addition,

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using Euler's telescoping lemma Bhatnagar derived in [3] an alternating version of (1) as follows:

$$\sum_{k=0}^{n}(-1)^k x^{n-k}\big(L_{k+1}+(x-2)F_k\big)=(-1)^n F_{n+1}. \tag{2}$$

On the other hand, Abd-Elhameed and Zeyada [1], without a reference to Edgar or Bhatnagar, derived two polynomial identities, one of which generalizes the polynomial identity of Sury [18]. Finally, in a recent paper, Chung, Yao and Zhou [4] provided extensions of Sury's relation and the alternating Sury's relation involving Fibonacci $k$-step and Lucas $k$-step polynomials.

This paper continues along the same path and introduces additional polynomial identities pertaining to generalized Lucas sequences. We delve into various aspects and examine previously unexplored special cases. Additionally, we provide proofs for the polynomial analogues of these identities for Chebyshev polynomials.

## 2. Additional polynomial generalizations

In this section, we give generalizations of (1) and (2) to the general Lucas sequences and prove two another identities of the same nature. For non-zero complex numbers $y$ and $z$, the Lucas sequences of the first kind, $(u_n(y,z))_{n\geq 0}$, and of the second kind, $(v_n(y,z))_{n\geq 0}$, are defined (see e.g., [14, Chapter 1]) through the recurrence relations

$$u_n(y,z)=yu_{n-1}(y,z)-zu_{n-2}(y,z),\quad n\geq 2,\quad u_0(y,z)=0,\ u_1(y,z)=1,$$

$$v_n(y,z)=yv_{n-1}(y,z)-zv_{n-2}(y,z),\quad n\geq 2,\quad v_0(y,z)=2,\ v_1(y,z)=y,$$

with

$$u_{-n}(y,z)=-u_n(y,z)z^{-n},\qquad v_{-n}(y,z)=v_n(y,z)z^{-n}.$$

Denote by $\tau$ and $\sigma$ the zeros of the characteristic polynomial $x^2-yx+z$ for the Lucas sequences. Then

$$\tau=\tau(y,z)=\frac{y+\sqrt{y^2-4z}}{2},\qquad \sigma=\sigma(y,z)=\frac{y-\sqrt{y^2-4z}}{2},$$

with $\tau+\sigma=y$, $\tau-\sigma=\sqrt{y^2-4z}$ and $\tau\sigma=z$. The difference equations are solved by the Binet formulas

$$u_n(\tau,\sigma)=\frac{\tau^n-\sigma^n}{\tau-\sigma},\qquad v_n(\tau,\sigma)=\tau^n+\sigma^n.$$

We need the following two lemmas, of which the first one contains basic telescoping summation identities.

**Lemma 2.1.** *If $(f_k)_{k\in\mathbb{Z}}$ is a real sequence and $j,\ n\in\mathbb{N}_0$ with $j\leq n$, then*

$$\sum_{k=j}^{n}(f_{k+1}-f_k)=f_{n+1}-f_j, \tag{3}$$

$$\sum_{k=j}^{n}(-1)^k(f_{k+1}+f_k)=(-1)^n f_{n+1}+(-1)^j f_j. \tag{4}$$

**Lemma 2.2.** *For all $x$, $y$, $r \in \mathbb{C}$, $r \neq 0$ and $k \in \mathbb{Z}$, we have*

$$\frac{x^k}{r^k}\big(rv_k(y,z) + (xy-2r)u_{k+1}(y,z)\big) = y\left(\frac{x^{k+1}}{r^k}u_{k+1}(y,z) - \frac{x^k}{r^{k-1}}u_k(y,z)\right), \tag{5}$$

$$\frac{x^k}{r^k}\big((y^2-4z)ru_k(y,z) + (xy-2r)v_{k+1}(y,z)\big) = y\left(\frac{x^{k+1}}{r^k}v_{k+1}(y,z) - \frac{x^k}{r^{k-1}}v_k(y,z)\right). \tag{6}$$

*Proof.* On account of the recurrence relation

$$u_{k+1}(y,z) + zu_{k-1}(y,z) = yu_k(y,z) \tag{7}$$

and the identity ([14, Identity (2.7)])

$$v_k(y,z) = u_{k+1}(y,z) - zu_{k-1}(y,z), \tag{8}$$

we have

$$rv_k(y,z) + (xy-2r)u_{k+1}(y,z) = y\big(xu_{k+1}(y,z) - ru_k(y,z)\big)$$

and hence (5). The proof of (6) is similar. We use

$$v_{k+1}(y,z) + zv_{k-1}(y,z) = yv_k(y,z) \tag{9}$$

and the identity

$$(y^2-4z)u_k(y,z) = v_{k+1}(y,z) - zv_{k-1}(y,z). \tag{10}$$

□

**Theorem 2.3.** *For all $x$, $y$, $r \in \mathbb{C}$ and $j$, $n \in \mathbb{N}_0$ with $j \leq n$,*

$$\sum_{k=j}^{n} r^{n-k}x^k\big(rv_k(y,z) + (xy-2r)u_{k+1}(y,z)\big) = x^{n+1}yu_{n+1}(y,z) - r^{n-j+1}x^j yu_j(y,z), \tag{11}$$

$$\sum_{k=j}^{n} r^{n-k}x^k\big((y^2-4z)ru_k(y,z) + (xy-2r)v_{k+1}(y,z)\big) = x^{n+1}yv_{n+1}(y,z) - r^{n-j+1}x^j yv_j(y,z). \tag{12}$$

*Proof.* To prove (11), sum (5) from $j$ to $n$, making use of (3). The proof of (12) is similar. □

**Remark 2.4.** *The equation* (11) *contains one result of Abd-Elhameed and Zeyada* [1, Theorem 1] *as a special case at $r = 1$ and $j = 0$. Actually, Abd-Elhameed and Zeyada stated their identity under the condition $x \neq 0$ (mutatis mutandis) but it is seen easily that it also holds for $x = 0$ as $u_0 = 0, u_1 = 1$ and $v_0 = 2$. Identity* (12) *seems to be new.*

**Lemma 2.5.** *For all $x$, $y$, $r \in \mathbb{C}$ and $k$, $n \in \mathbb{Z}$, we have*

$$x^{n-k}r^k\big(rv_{k+1}(y,z) + (xy+2rz)u_k(y,z)\big) = y\big(r^{k+1}x^{n-k}u_{k+1}(y,z) + r^k x^{n-k+1}u_k(y,z)\big), \tag{13}$$

$$x^{n-k}r^k\big((y^2-4z)ru_{k+1}(y,z) + (xy+2rz)v_k(y,z)\big) = y\big(r^{k+1}x^{n-k}v_{k+1}(y,z) + r^k x^{n-k+1}v_k(y,z)\big). \tag{14}$$

*Proof.* Eliminating $u_{k+1}(y,z)$ between (7) and (8) gives

$$v_k(y,z) = yu_k(y,z) - 2zu_{k-1}(y,z)$$

and hence

$$rv_k(y,z) + (xy+2rz)u_{k-1}(y,z) = ryu_k(y,z) + xyu_{k-1}(y,z),$$

from which (13) follows. The proof of (14) is similar; eliminate $v_{k+1}(y,z)$ between (9) and (10). ☐

**Theorem 2.6.** *For all $x$, $y$, $r \in \mathbb{C}$ and $j$, $n \in \mathbb{N}_0$ with $j \le n$,*

$$\sum_{k=j}^{n}(-1)^k x^{n-k}r^k\big(rv_{k+1}(y,z) + (xy+2rz)u_k(y,z)\big) = (-1)^n yr^{n+1}u_{n+1}(y,z) + (-1)^j r^j yx^{n-j+1}u_j(y,z), \tag{15}$$

$$\sum_{k=j}^{n}(-1)^k x^{n-k}r^k\,\big((y^2-4z)ru_{k+1}(y,z) + (xy+2rz)v_k(y,z)\big) = (-1)^n yr^{n+1}v_{n+1}(y,z) + (-1)^j r^j yx^{n-j+1}v_j(y,z). \tag{16}$$

*Proof.* Sum each of (13) and (14) from $j$ to $n$, making use of (4) since the right-hand side telescopes in each case. ☐

**Remark 2.7.** *Identity* (15) *is an additional polynomial generalization of identity* (2). *Identity* (16), *its Lucas version, is also presumably new.*

## 3. Applications to Fibonacci and Lucas polynomials

This section is devoted to discussing in detail some special cases of Theorems 2.3 and 2.6, thereby paying special attention to Fibonacci and Lucas polynomials $F_n(x)$ and $L_n(x)$. The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, where $F_0(x) = 0$, $F_1(x) = x$, and $n \ge 2$. The polynomials $L_n(x)$ are defined by $L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$, where $L_0(x) = 2$, $L_1(x) = x$, and $n \ge 2$.

The polynomial identities presented in Proposition 3.1 below are also special cases of Theorem 1 in [4]. We think, however, that they deserve to be discussed. In the course of discussion we will rediscover some identities from [1] but also present new relations that we did not find in the literature and which deserve recognition.

First we note that $F_n(x) = u_n(x,-1)$, $L_n(x) = v_n(x,-1)$ and $F_n(1) = F_n$, $L_n(1) = L_n$. Also, $F_n(2) = P_n$ and $L_n(2) = Q_n$, where $P_n$ and $Q_n$ denote the Pell and Pell–Lucas numbers, respectively. Other special values of these polynomials will be used later.

**Corollary 3.1.** *For all $x, y \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n} x^k \big(L_k(y) + (xy-2)F_{k+1}(y)\big) = x^{n+1} y F_{n+1}(y), \tag{17}$$

$$\sum_{k=0}^{n} x^k \big((y^2+4)F_k(y) + (xy-2)L_{k+1}(y)\big) = y\big(x^{n+1}L_{n+1}(y) - 2\big). \tag{18}$$

*Proof.* Set $z = -1$, $r = 1$ and $j = 0$ in (11) and (12). □

Corollary 3.1 provides two polynomial extensions of the identity (1). Both identities connect interestingly Fibonacci and Lucas polynomials. Since $L_n(y) = F_{n-1}(y) + F_{n+1}(y)$ and $(y^2+4)F_n(y) = L_{n-1}(y) + L_{n+1}(y)$ (see [8, Chapters 37 and 38]) they may also be disconnected:

$$\sum_{k=0}^{n} x^k \big(xF_{k+1}(y) - F_k(y)\big) = x^{n+1} F_{n+1}(y)$$

and

$$\sum_{k=0}^{n} x^k \big(xL_{k+1}(y) - L_k(y)\big) = x^{n+1} L_{n+1}(y) - xy.$$

**Corollary 3.2.** *For all $x \in \mathbb{C}$ and $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} x^k \big(L_k + (x-2)F_{k+1}\big) = x^{n+1} F_{n+1}, \tag{19}$$

$$\sum_{k=0}^{n} x^k \big(5F_k + (x-2)L_{k+1}\big) = x^{n+1} L_{n+1} - 2,$$

$$\sum_{k=0}^{n} x^k \big(Q_k + 2(x-1)P_{k+1}\big) = 2x^{n+1} P_{n+1}, \tag{20}$$

$$\sum_{k=0}^{n} x^k \big(4P_k + (x-1)Q_{k+1}\big) = x^{n+1} Q_{n+1} - 2.$$

*Proof.* Set $y = 1$ and $y = 2$, in turn, in (17) and (18). □

**Remark 3.3.** *We point out that* (19) *and* (20) *are not new. These are Equations (11) and (15) in* [1].

**Corollary 3.4.** *For all $x \in \mathbb{C}$ and $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} x^k \big(L_{3k} + (2x-1)F_{3k+3}\big) = 2x^{n+1} F_{3n+3},$$

$$\sum_{k=0}^{n} x^k \big(5F_{3k} + (2x-1)L_{3k+3}\big) = 2x^{n+1} L_{3n+3} - 4.$$

*Proof.* Set $y = 4$ in (17) and (18), respectively, and use the evaluations $F_n(4) = F_{3n}/2$ (see [19]) as well as $L_n(4) = L_{3n}$. □

**Corollary 3.5.** *For all $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} 2^k x^{n-k} L_k(x) = 2^{n+1} F_{n+1}(x), \quad x \in \mathbb{C},$$

$$\sum_{k=0}^{n} 2^k x^{n-k} F_k(x) = \frac{2}{x^2+4}\big(2^n L_{n+1}(x) - x^{n+1}\big), \quad x \in \mathbb{C} \setminus \{\pm 2i\}.$$

*Proof.* Interchange $x$ and $y$ in each of (17) and (18) and set $y = 2/x$. □

The results in Corollaries 3.6 – 3.9 are identities involving only even-indexed Fibonacci and Lucas polynomials.

**Corollary 3.6.** *For all $x, y \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n} x^k \big(yL_{2k}(y) + (x(y^2+2)-2)F_{2k+2}(y)\big) = x^{n+1}(y^2+2)F_{2n+2}(y), \tag{21}$$

$$\begin{aligned}\sum_{k=0}^{n} x^k \big(y(y^2+4)F_{2k}(y) + \big(x(y^2+2)-2\big)L_{2k+2}(y)\big) \\ = (y^2+2)\big(x^{n+1}L_{2n+2}(y) - 2\big).\end{aligned} \tag{22}$$

*Proof.* We only prove (21). Use Corollary 3.1 and relations

$$L_n\big(i(x^2+2)\big) = i^n L_{2n}(x) \quad \text{and} \quad F_n\big(i(x^2+2)\big) = i^{n-1}\frac{F_{2n}(x)}{x},$$

where $i = \sqrt{-1}$. These relations are proved easily by inserting the respective values into the Binet forms. When simplifying replace $x$ by $x/i$. The other proof is similar. □

**Corollary 3.7.** *For all $x \in \mathbb{C}$ and $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} x^k \big(L_{2k} + (3x-2)F_{2k+2}\big) = 3x^{n+1}F_{2n+2},$$

$$\sum_{k=0}^{n} x^k \big(5F_{2k} + (3x-2)L_{2k+2}\big) = 3\big(x^{n+1}L_{2n+2} - 2\big),$$

$$\sum_{k=0}^{n} x^k \big(Q_{2k} + (3x-1)P_{2k+2}\big) = 3x^{n+1}P_{2n+2},$$

$$\sum_{k=0}^{n} x^k \big(8P_{2k} + (3x-1)Q_{2k+2}\big) = 3\big(x^{n+1}Q_{2n+2} - 2\big).$$

*Proof.* Set $y = 1$ and $y = 2$ in (21) and (22), respectively. □

**Corollary 3.8.** *For all $x \in \mathbb{C}$ and $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} x^k \big(3L_{4k} + (7x-2)F_{4k+4}\big) = 7x^{n+1}F_{4n+4},$$

$$\sum_{k=0}^{n} x^k \big(15F_{4k} + (7x-2)L_{4k+4}\big) = 7\big(x^{n+1}L_{4n+4} - 2\big).$$

*Proof.* Set $y = \sqrt{5}$ in (21) and (22), respectively. Then, utilize the evaluations $F_{2n}(\sqrt{5}) = (\sqrt{5}/3)F_{4n}$ and $L_{2n}(\sqrt{5}) = L_{4n}$, which can be easily proven by substituting the respective values into the corresponding Binet form. □

**Corollary 3.9.** *For all $x \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} \Big(\frac{x^2+2}{2}\Big)^{n-k} L_{2k}(x) = \frac{2}{x} F_{2n+2}(x),$$

$$\sum_{k=0}^{n} \Big(\frac{x^2+2}{2}\Big)^{n-k} F_{2k}(x) = \frac{2}{x(x^2+4)}\Big(L_{2n+2}(x) - 2\Big(\frac{x^2+2}{2}\Big)^{n+1}\Big).$$

*Proof.* Set $x = \frac{2}{y^2+2}$ in (21) and in (22) and replace $y$ by $x$. □

The next identities provide still other interesting relations that nicely generalize and complement Edgar–Bhatnagar identity.

**Proposition 3.10.** *For all $x \in \mathbb{C}$ and integers $m$ and $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n} x^k \big(F_m L_{mk} + (xL_m - 2)F_{m(k+1)}\big) = x^{n+1} L_m F_{m(n+1)}, \tag{23}$$

$$\sum_{k=0}^{n} x^k \big(5F_m F_{mk} + (xL_m - 2)L_{m(k+1)}\big) = L_m\big(x^{n+1} L_{m(n+1)} - 2\big).$$

*Proof.* We only prove (23). Let $m$ be odd. Then by straightforward calculation $L_k(L_m) = L_{mk}$ and $F_k(L_m) = F_{mk}/F_m$, where we used the identity $L_n^2 = 5F_n^2 + (-1)^n 4$. This immediately gives the first identity, using (17). Let $m$ be even now. Then we work with $iL_m$ as an argument and get $L_k(iL_m) = i^k L_{mk}$ and $F_k(iL_m) = i^{k-1} F_{mk}/F_m$. Inserting these results into (17) and finally replacing $x$ by $x/i$ completes the proof. □

We proceed with a (short) discussion of Theorem 2.6. Making the choices $z = -1$, $r = 1$ and $j = 0$ it becomes the next identities for Fibonacci and Lucas polynomials, which can also be obtained from [4, Theorem 1].

**Proposition 3.11.** *For all $x$, $y \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n} (-1)^{n-k} x^{n-k} \big(L_{k+1}(y) + (xy-2)F_k(y)\big) = yF_{n+1}(y), \tag{24}$$

$$\sum_{k=0}^{n} (-1)^{n-k} x^{n-k} \big((y^2+4)F_{k+1}(y) + (xy-2)L_k(y)\big) = yL_{n+1}(y) + 2(-1)^n y x^{n+1}. \tag{25}$$

**Corollary 3.12.** *For all $x \in \mathbb{C}$ and $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{n} (-1)^{n-k} x^{n-k} \big(L_{k+1} + (x-2)F_k\big) = F_{n+1},$$

$$\sum_{k=0}^{n} (-1)^{n-k} x^{n-k} \big(5F_{k+1} + (x-2)L_k\big) = L_{n+1} + 2(-1)^n x^{n+1},$$

$$\sum_{k=0}^{n}(-1)^{n-k}x^{n-k}\big(Q_{k+1}+2(x-1)P_k\big)=2P_{n+1},$$
$$\sum_{k=0}^{n}(-1)^{n-k}x^{n-k}\big(4P_{k+1}+(x-1)Q_k\big)=Q_{n+1}+2(-1)^n x^{n+1}.$$

*Proof.* Set $y=1$ and $y=2$ in (24) and (25), respectively. □

The first identity in the corollary above was derived in [11].

We are not interested in repeating the analogous results to Corollaries (3.4)–(3.9). We mention, however, that the symmetry gets lost when passing from Proposition 3.10 to its alternating analogue.

**Proposition 3.13.** *For all $x\in\mathbb{C}$ and integers $m$, and $n\in\mathbb{N}_0$, we have*

$$\sum_{k=0}^{n}(-1)^{n-k}x^{n-k}\big(F_mL_{m(k+1)}+(xL_m-2)F_{mk}\big)=L_mF_{m(n+1)},\quad m\ \textit{odd},$$
$$\sum_{k=0}^{n}x^{n-k}\big(F_mL_{m(k+1)}-(xL_m-2)F_{mk}\big)=L_mF_{m(n+1)},\quad m\ \textit{even}.$$

*Similarly, we have*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}x^{n-k}&\big(5F_mF_{m(k+1)}+(xL_m-2)L_{mk}\big)\\&=L_m\big(L_{m(n+1)}+2(-1)^n x^{n+1}\big),\quad m\ \textit{odd},\end{aligned}$$
$$\sum_{k=0}^{n}x^{n-k}\big(5F_mF_{m(k+1)}-(xL_m-2)L_{mk}\big)=L_m\big(L_{m(n+1)}-2x^{n+1}\big),\quad m\ \textit{even}.$$

## 4. Extension to Chebyshev polynomials

Let $T_n(y)$ and $U_n(y)$ be the Chebyshev polynomials of the first and second kind defined by $T_0(y)=1$, $T_1(y)=y$ and for $n\geq 2$

$$T_n(y)=2yT_{n-1}(y)-T_{n-2}(y) \tag{26}$$

and $U_0(y)=1$, $U_1(y)=2y$ and for $n\geq 2$

$$U_n(y)=2yU_{n-1}(y)-U_{n-2}(y). \tag{27}$$

Both sequences $(T_n(y))_{n\geq 0}$ and $(U_n(y))_{n\geq 0}$ can be extended to negative subscripts by writing $T_{-n}(y)=T_n(y)$ and $U_{-n}(y)=2yU_{-(n-1)}(y)-U_{-(n-2)}(y)$. See [12] among others for further information on these polynomials.

**Lemma 4.1.** *For all $x$, $y$, $r\in\mathbb{C}$, $r\neq 0$ and $k\in\mathbb{Z}$, we have*

$$\frac{x^k}{r^k}\big(rT_k(y)+(xy-r)U_k(y)\big)=y\left(\frac{x^{k+1}}{r^k}U_k(y)-\frac{x^k}{r^{k-1}}U_{k-1}(y)\right), \tag{28}$$

*and*

$$\begin{aligned}&\frac{x^k}{r^k}\left((y^2-1)rU_{k-2}(y)+(xy-r)T_k(y)\right)\\&\qquad=y\left(\frac{x^{k+1}}{r^k}T_k(y)-\frac{x^k}{r^{k-1}}T_{k-1}(y)\right).\end{aligned} \tag{29}$$

*Proof.* In view of the recurrence relation (27) and the identity $2T_k(y) = U_k(y) - U_{k-2}(y)$ [12, Chapter 1], we have

$$rT_k(y) + (xy - r)U_k(y) = y\big(xU_k(y) - rU_{k-1}(y)\big),$$

from which (28) follows. Similarly, using the recurrence relation (26) and the identity $2(y^2 - 1)U_{k-2}(y) = T_k(y) - T_{k-2}(y)$ [12, Chapter 2] we obtain

$$(y^2 - 1)rU_{k-2}(y) + (xy - r)T_k(y) = y\big(xT_k(y) - rT_{k-1}(y)\big)$$

from which (29) follows. □

**Theorem 4.2.** *For all $x, y, r \in \mathbb{C}$ and $j, n \in \mathbb{N}_0$ with $j \le n$, we have*

$$\sum_{k=j}^{n} r^{n-k}x^k\big(rT_k(y) + (xy - r)U_k(y)\big) = x^{n+1}yU_n(y) - r^{n-j+1}x^jyU_{j-1}(y),$$

$$\sum_{k=j}^{n} r^{n-k}x^k\left((y^2 - 1)rU_{k-2}(y) + (xy - r)T_k(y)\right) = x^{n+1}yT_n(y) - r^{n-j+1}x^jyT_{j-1}(y).$$

*In particular, we have*

$$\sum_{k=0}^{n} x^k\big(T_k(y) + (xy - 1)U_k(y)\big) = x^{n+1}yU_n(y),$$

*as well as*

$$\sum_{k=0}^{n} x^k\big((y^2 - 1)U_{k-2}(y) + (xy - 1)T_k(y)\big) = x^{n+1}yT_n(y) - y^2.$$

*Proof.* Sum each of (28) and (29), using (3). □

**Lemma 4.3.** *For all $x \in \mathbb{C}$, and $n \in \mathbb{N}_0$, we have*

$$U_n\Big(\frac{x^2+2}{2}\Big) = (-1)^{n+1}\frac{i}{x}U_{2n+1}\Big(\frac{ix}{2}\Big), \quad x \ne 0,$$

$$T_n\Big(\frac{x^2+2}{2}\Big) = (-1)^nT_{2n}\Big(\frac{ix}{2}\Big).$$

*Proof.* Both identities can be proved by induction on $n$. □

**Corollary 4.4.** *For all nonzero $x, y \in \mathbb{C}$ and all $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n}(-1)^{n-k}x^k\Big(2yT_{2k}(y) + \big(x(1 - 2y^2) - 1\big)U_{2k+1}(y)\Big) = x^{n+1}(1 - 2y^2)U_{2n+1}(y),$$

$$\sum_{k=0}^{n}(-1)^{n-k}x^k\Big(2y(y^2 - 1)U_{2k-3}(y) + \big(x(1 - 2y^2) - 1\big)T_{2k}(y)\Big) = x^{n+1}(1 - 2y^2)T_{2n}(y) - (-1)^n(1 - 2y^2)^2$$

*with $U_{-3}(y) = -2y, U_{-2}(y) = -1$ and $U_{-1}(y) = 0$.*

*Proof.* Combine Theorem 4.2 with Lemma 4.3 and simplify. □

## 5. Conclusion

In this note we have provided various complements to Fibonacci–Lucas identities first proved by Edgar, Sury, and Bhatnagar [3, 7, 18]. Starting with additional polynomial generalizations, we have also presented a detailed discussion of charming identities for Fibonacci and Lucas polynomials as special cases. Finally, we have proved the analogues for Chebyshev polynomials. We conclude noting two observations. First, concerning the results for Fibonacci and Lucas polynomials we remark that our results can be combined with the beautiful identities derived by Seiffert [15, 16, 17] to get some nontrivial sum relations. This is left as a possible future research project. Second, as it was shown here, the identities introduced and discussed are not limited to Fibonacci and Lucas, and Chebyshev polynomials. Analogues are possible for Pell polynomials, Jacobsthal polynomials and others. By way of a final illustration, we state the identities in Theorem 2.3 for the Jacobsthal and Jacobsthal–Lucas sequences $(J_n) = (u_n(1, -2))$ and $(j_n) = (v_n(1, -2))$: For all $x$, $r \in \mathbb{C}$ and $s$, $n \in \mathbb{N}_0$ with $s \leq n$, we have

$$\sum_{k=s}^{n} r^{n-k} x^k \big(r j_k + (x-2r) J_{k+1}\big) = x^{n+1} J_{n+1} - r^{n-s+1} x^s J_s,$$

$$\sum_{k=s}^{n} r^{n-k} x^k \big(9r J_k + (x-2r) j_{k+1}\big) = x^{n+1} j_{n+1} - r^{n-s+1} x^s j_s.$$

Interesting identities can be drawn from these relations. We leave them for the interested readers.

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